

GLOBAL EXISTENCE OF THE TWO-DIMENSIONAL QGE WITH SUB-CRITICAL DISSIPATION

JAMEL BENAMEUR AND MOEZ BENHAMED

ABSTRACT. In this paper, we study the sub-critical dissipative quasi-geostrophic equations (\mathbf{S}_α) . We prove that there exists a unique local-in-time solution for any large initial data θ_0 in the space $\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$ defined by (1.1). Moreover we show that (\mathbf{S}_α) has a global solution in time if the norms of the initial data in $\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$ are bounded by $1/4$. Also, we prove a blow-up criterion of the non global solution of (\mathbf{S}_α) .

CONTENTS

| | |
|--|----|
| 1. Introduction | 1 |
| 2. Notations and Preliminaries results | 3 |
| 2.1. Notations | 3 |
| 2.2. Preliminaries results | 3 |
| 3. Proof of Theorem 1.1 | 7 |
| 3.1. Existence | 7 |
| 3.2. Uniqueness | 11 |
| 3.3. Small initial data | 13 |
| 4. Proof of Theorem 1.2 | 14 |
| References | 15 |

1. INTRODUCTION

In this article, we study the initial value-problem for the two-dimensional quasi-geostrophic equation with sub-critical dissipation (\mathbf{S}_α) , $1/2 < \alpha \leq 1$,

$$(\mathbf{S}_\alpha) \quad \begin{cases} \partial_t \theta + (-\Delta)^\alpha \theta + u \cdot \nabla \theta = 0 \\ \theta(0) = \theta_0 \end{cases}$$

where $1/2 < \alpha \leq 1$ is a real number. The variable θ represents potential temperature, $u = (\partial_2(-\Delta)^{-1/2}\theta, -\partial_1(-\Delta)^{-1/2}\theta)$ is the fluid velocity. In the following, we are interested in the case when $1/2 < \alpha \leq 1$ as the case $\alpha = 1/2$ is very much similar to the 2D Navier-Stokes which is known to be a wellposed problem, and therefore the case $\alpha > 1/2$ is even easier to deal with.

The mathematical study of the non-dissipative case has first been proposed by Constantin, Majda and Tabak in [13] where it is shown to be an analogue to the 3D Euler equations. The dissipative case has then been studied by Constantin and Wu in [11] when $\alpha > 1/2$

Date: August 26, 2014.

2000 Mathematics Subject Classification. 35A35; 35B05; 35B40; 35Q55; 81Q20.

Key words and phrases. Surface quasi-geostrophic equations, sub-critical, Critical spaces, Global existence and uniqueness.

and global existence in Sobolev spaces is studied by Constantin.

In this paper, we study (\mathbf{S}_α) in scaling invariant spaces. Solvability of evolution equations in scaling invariant spaces is well-developed in the context of the Navier-Stokes equations. For example, if we restrict the function spaces to the energy spaces, the optimal result is due to Fujita and Kato in [7]. Later, Chemin [10] proved similar results in the framework of Besov spaces $\dot{B}_{p,q}^{\frac{2}{p}-1}$. Let us find the scaling invariant critical spaces for (\mathbf{S}_α) . The equation is invariant under the following scaling:

$$\theta_\lambda(t, x) = \lambda^{2\alpha-1} \theta(\lambda^{2\alpha} t, \lambda x), \quad \text{with initial data } \theta_\lambda(0, x) = \theta_\lambda^0(x) = \lambda^{2\alpha-1} \theta^0(\lambda x).$$

So $L^{\frac{2}{2\alpha-1}}, \dot{H}^{2-2\alpha}$ and $\dot{B}_{p,q}^{1+\frac{2}{p}-2\alpha}$ are critical spaces. The global well-posedness for small initial data in critical Besov spaces with $p < \infty$ was obtained in [4, 14]. The global well-posedness for large data in critical spaces was obtained by several authors; in Lebesgue space $L^{\frac{2}{2\alpha-1}}$ by Carrillo and Ferreira [9] for $\alpha > \frac{1}{2}$, in energy space H^1 by Dong and Du [5], and in Besov spaces by Abidi and Hmidi [1], Hmidi and Keraani [2] and Wang and Zhang [15] with $p = \infty$.

In this paper, we will solve the system (\mathbf{S}_α) in the critical space by means of a contraction argument. Thus, we can obtain a unique local solution to the system (\mathbf{S}_α) for any initial data in the critical space and prove the corresponding solution will be global if the initial data is sufficient small. Before giving our main result let us first precise the notion of critical space.

For $\sigma \in \mathbb{R}$, we define the functional space

$$(1.1) \quad \mathcal{X}^\sigma(\mathbb{R}^2) = \left\{ f \in \mathcal{D}'(\mathbb{R}^2); (\xi \mapsto |\xi|^\sigma \widehat{f}(\xi)) \in L^1(\mathbb{R}^2) \right\}$$

which is equipped with the norm

$$\|f\|_{\mathcal{X}^\sigma} = \int_{\mathbb{R}^2} |\xi|^\sigma |\widehat{f}(\xi)| d\xi.$$

Now we give our first result overcoming the above mentioned smallness assumption. Our global existence result reads as follows.

Theorem 1.1. *Let $\theta^0 \in \mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$. There is a time $T > 0$ and unique solution $\theta \in \mathcal{C}([0, T], \mathcal{X}^{1-2\alpha}(\mathbb{R}^2))$ of (\mathbf{S}_α) , moreover $\theta \in L^1([0, T], \mathcal{X}^1(\mathbb{R}^2))$. If $\|\theta^0\|_{\mathcal{X}^{1-2\alpha}} < 1/4$, there are global existence and*

$$(1.2) \quad \|\theta\|_{\mathcal{X}^{1-2\alpha}} + (1 - 4\|\theta^0\|_{\mathcal{X}^{1-2\alpha}}) \int_0^t \|\theta\|_{\mathcal{X}^1} \leq \|\theta^0\|_{\mathcal{X}^{1-2\alpha}}, \quad \forall t \geq 0.$$

The proof of the above theorem is based on Fixed Point Theorem. We will establish a local existence result and will be able to get global existence that is essentially based on energy estimate of the solution in the space $\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$.

We state now our second main result.

Theorem 1.2. *Let $\theta \in \mathcal{C}([0, T^*), \mathcal{X}^{1-2\alpha}(\mathbb{R}^2))$ be the maximal solution of (\mathbf{S}_α) given by Theorem 1.1. Then*

$$(1.3) \quad T^* < \infty \implies \int_0^{T^*} \|\theta\|_{\mathcal{X}^1} = \infty.$$

The rest of this paper is divided into three sections. Section 2 is further divided into two subsections. Section 2.1 recalls the notations, Section 2.2 preliminaries results. Section 3 is further divided into three subsections. Section 3.1 deals with the existence of solution with any large initial data in the spaces in $\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$, section 3.2 deals with the uniqueness of solution, and section 3.3 is devoted to global existence. The section 4 proves a blow-up criterion of the non global solution given by Theorem 1.1.

2. NOTATIONS AND PRELIMINARIES RESULTS

2.1. Notations. In this short paragraph, we give some notations

- For f , we denote u_f the following

$$u_f = (\partial_2(-\Delta)^{-1/2}f, -\partial_1(-\Delta)^{-1/2}f)$$

- The Fourier transform $\mathcal{F}(f)$ of a tempered distribution f on \mathbb{R}^2 is defined as

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^2} e^{-ix\xi} f(x) dx.$$

- The inverse Fourier formula is

$$\mathcal{F}^{-1}(f)(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix\xi} f(x) dx.$$

- For any Banach space $(B, \|\cdot\|)$, any real number $1 \leq p \leq \infty$ and any time $T > 0$, we will denote by $L_T^p(B)$ the space of all measurable functions $t \in [0, T] \mapsto f(t) \in B$ such that $(t \mapsto \|f(t)\|) \in L^p([0, T])$.

- The fractional Laplacian operator $(-\Delta)^\alpha$ for a real number α is defined through the Fourier transform, namely

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi).$$

2.2. Preliminaries results. The main result of this section is the following lemmas that will play a crucial role in the proof of our main theorem.

Lemma 2.1. *Let $\alpha \in]\frac{1}{2}, 1]$. We have the following inequalities*

$$(2.1) \quad \|f\|_{\mathcal{X}^{2-2\alpha}} \leq \|f\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}} \|f\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}}$$

and

$$(2.2) \quad \|f\|_{\mathcal{X}^0} \leq \|f\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}} \|f\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}}.$$

Proof.

- **Proof of (2.1).**

By choosing $(p, q) = (2\alpha, \frac{2\alpha}{2\alpha-1})$ and applying the Hölder's Inequality, we deduce that

$$\begin{aligned} \|f\|_{\mathcal{X}^{2-2\sigma}} &= \int_{\xi} |\xi|^{2-2\sigma} |\widehat{f}(\xi)| d\xi \\ &\leq \int_{\xi} |\xi|^{\frac{1}{2\alpha}} |\widehat{f}(\xi)|^{\frac{1}{2\alpha}} |\xi|^{\frac{2\alpha-1}{2\alpha}} |\widehat{f}(\xi)|^{\frac{2\alpha-1}{2\alpha}} |\xi|^{1-2\alpha} d\xi \\ &\leq \left(\int_{\xi} |\xi| |\widehat{f}(\xi)| d\xi \right)^{1/2\alpha} \left(\int_{\xi} |\xi|^{1-2\sigma} |\widehat{f}(\xi)| d\xi \right)^{1-\frac{1}{2\alpha}} \end{aligned}$$

Hence,

$$\|f\|_{\mathcal{X}^{2-2\sigma}} \leq \|f\|_{\mathcal{X}^1}^{1/2\alpha} \|f\|_{\mathcal{X}^{1-2\sigma}}^{1-\frac{1}{2\alpha}}.$$

• **Proof of (2.2).**

To estimate $\|f\|_{\mathcal{X}^0}$, we choose $p = 2\alpha$ and $q = \frac{2\alpha}{2\alpha-1}$ and apply the Hölder's Inequalities. We obtain,

$$\begin{aligned} \|f\|_{\mathcal{X}^0} &= \int_{\xi} |\xi|^0 |\widehat{f}(\xi)| d\xi \\ &\leq \int_{\xi} |\xi|^{\frac{2\alpha-1}{2\alpha}} |\widehat{f}(\xi)|^{\frac{2\alpha-1}{2\alpha}} |\xi|^{\frac{1-2\alpha}{2\alpha}} |\widehat{f}(\xi)|^{\frac{1}{2\alpha}} d\xi \\ &\leq \left(\int_{\xi} |\xi| |\widehat{f}(\xi)| d\xi \right)^{1/2\alpha} \left(\int_{\xi} |\xi|^{1-2\alpha} |\widehat{f}(\xi)| d\xi \right)^{1-\frac{1}{2\alpha}} \end{aligned}$$

Hence,

$$\|f\|_{\mathcal{X}^0} \leq \|f\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}} \|f\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}}.$$

The following lemma, which is a direct consequence of the preceding one, will be useful in the proof of Theorem 1.1.

Lemma 2.2. *Let $f, g \in \mathcal{X}^{1-2\alpha}(\mathbb{R}^2) \cap \mathcal{X}^1(\mathbb{R}^2)$. We have the following inequalities*

$$(2.3) \quad \|fg\|_{\mathcal{X}^{2-2\alpha}} \leq 2\|f\|_{\mathcal{X}^{2-2\alpha}} \|g\|_{\mathcal{X}^0} + 2\|f\|_{\mathcal{X}^0} \|g\|_{\mathcal{X}^{2-2\alpha}}.$$

$$(2.4) \quad \begin{aligned} \|fg\|_{\mathcal{X}^{2-2\alpha}} &\leq 2\|f\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}} \|f\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}} \|g\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}} \|g\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}} \\ &\quad + 2\|f\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}} \|f\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}} \|g\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}} \|g\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}}. \end{aligned}$$

Particularly

$$(2.5) \quad \|f \cdot f\|_{\mathcal{X}^{2-2\alpha}} \leq 4\|f\|_{\mathcal{X}^{1-2\alpha}} \|f\|_{\mathcal{X}^1}.$$

Proof.

• **Proof of (2.3).**

$$\begin{aligned} \|fg\|_{\mathcal{X}^{2-2\alpha}} &= \int_{\xi} |\xi|^{2-2\alpha} |\widehat{fg}(\xi)| d\xi \\ &= \int_{\xi} |\xi|^{2-2\alpha} |\widehat{f} * \widehat{g}(\xi)| d\xi \\ &= \int_{\xi} |\xi|^{2-2\alpha} \left| \int_{\eta} \widehat{f}(\eta) \widehat{g}(\xi - \eta) d\eta \right| d\xi. \end{aligned}$$

Then

$$\begin{aligned} \|fg\|_{\mathcal{X}^{2-2\alpha}} &\leq \int_{\xi} \int_{\eta} |\xi|^{2-2\alpha} |\widehat{f}(\eta)| |\widehat{g}(\xi - \eta)| d\xi \\ &\leq \int_{\xi} |\xi|^{2-2\alpha} \int_{\eta} |\widehat{f}(\eta)| |\widehat{g}(\xi - \eta)| d\xi \\ &\leq I_1 + I_2. \end{aligned}$$

Where,

$$\begin{aligned} I_1 &:= \int_{\xi} |\xi|^{2-2\alpha} \int_{|\eta| < |\xi-\eta|} |\widehat{f}(\eta)| |\widehat{g}(\xi-\eta)| d\xi \\ I_2 &:= \int_{\xi} |\xi|^{2-2\alpha} \int_{|\eta| > |\xi-\eta|} |\widehat{f}(\eta)| |\widehat{g}(\xi-\eta)| d\xi. \end{aligned}$$

On the one hand, if $|\eta| < |\xi - \eta|$, we have

$$|\xi| \leq |\xi - \eta| + |\eta| \leq 2 \max(|\xi - \eta|, |\eta|) = 2|\xi - \eta|.$$

Then

$$\begin{aligned} |\xi|^{2-2\alpha} &\leq 2^{2-2\alpha} |\xi - \eta|^{2-2\alpha} \\ &\leq 2 |\xi - \eta|^{2-2\alpha}, \quad \text{for all } \alpha \in (1/2, 1]. \end{aligned}$$

Therefore

$$\begin{aligned} I_1 &= \int_{\xi} |\xi|^{2-2\alpha} \int_{|\eta| < |\xi-\eta|} |\widehat{f}(\eta)| |\widehat{g}(\xi-\eta)| d\xi \\ &\leq 2 \int_{\xi} \int_{\eta} |\widehat{f}(\eta)| |\xi - \eta|^{2-2\alpha} |\widehat{g}(\xi - \eta)| d\xi \\ &\leq 2 \|\widehat{f}\| * |\xi|^{2-2\alpha} \|\widehat{g}\|_{L^1}. \end{aligned}$$

By Young inequality, we get

$$(2.6) \quad I_1 \leq 2 \|f\|_{\mathcal{X}^0} \|g\|_{\mathcal{X}^{2-2\alpha}}.$$

On the other hand, if $|\eta| > |\xi - \eta|$, we have

$$|\xi| \leq |\xi - \eta| + |\eta| \leq 2 \max(|\xi - \eta|, |\eta|) = 2|\eta|.$$

Then

$$\begin{aligned} |\xi|^{2-2\alpha} &\leq 2^{2-2\alpha} |\eta|^{2-2\alpha} \\ &\leq 2 |\eta|^{2-2\alpha}, \quad \text{for all } \alpha \in (1/2, 1]. \end{aligned}$$

Hence

$$\begin{aligned} I_2 &= \int_{\xi} |\xi|^{2-2\alpha} \int_{|\eta| > |\xi-\eta|} |\widehat{f}(\eta)| |\widehat{g}(\xi-\eta)| d\xi \\ &\leq 2 \int_{\xi} \int_{\eta} |\eta|^{2-2\alpha} |\widehat{f}(\eta)| |\widehat{g}(\xi-\eta)| d\xi \\ &\leq 2 \|\xi\|^{2-2\alpha} \|\widehat{f}\| * \|\widehat{g}\|_{L^1}. \end{aligned}$$

Young inequality gives

$$(2.7) \quad I_2 \leq 2 \|f\|_{\mathcal{X}^{2-2\alpha}} \|g\|_{\mathcal{X}^0}.$$

Combining (2.6) and (2.7), we get (2.3).

• Proof of (2.4).

By including the inequalities (2.1) and (2.2) in (2.3), we obtain

$$\|fg\|_{\mathcal{X}^{2-2\alpha}} \leq 2 \|f\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}} \|f\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}} \|g\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}} \|g\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}} + 2 \|f\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}} \|f\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}} \|g\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}} \|g\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}}.$$

The proof of Theorem 1.1 requires the following lemma.

Lemma 2.3. *Let $\theta \in L_T^\infty(\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)) \cap L_T^1(\mathcal{X}^1(\mathbb{R}^2))$. Then*

$$\left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} \operatorname{div}(\theta u_\theta) dz \right\|_{\mathcal{X}^{1-2\alpha}} \leq C_\alpha \|\theta\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})} \|\theta\|_{L_T^1(\mathcal{X}^1)}.$$

Proof. We have

$$\begin{aligned} \left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} \operatorname{div}(\theta u_\theta) dz \right\|_{\mathcal{X}^{1-2\alpha}} &\leq \int_0^t \|e^{-(t-z)|D|^{2\alpha}} \operatorname{div}(\theta u_\theta)\|_{\mathcal{X}^{1-2\alpha}} dz \\ &\leq \int_0^t \|e^{-(t-z)|D|^{2\alpha}}(\theta u_\theta)\|_{\mathcal{X}^{2-2\alpha}} dz \\ &\leq \int_0^t \|\theta u_\theta\|_{\mathcal{X}^{2-2\alpha}} dz. \end{aligned}$$

Using Lemma 2.2, we obtain,

$$\left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} \operatorname{div}(\theta u_\theta) dz \right\|_{\mathcal{X}^{1-2\alpha}} \leq 2 \int_0^t \|\theta\|_{\mathcal{X}^{2-2\alpha}} \|u_\theta\|_{\mathcal{X}^0} + \|\theta\|_{\mathcal{X}^0} \|u_\theta\|_{\mathcal{X}^{2-2\alpha}} dz.$$

Using Lemma 2.3 and the fact $\|u_\theta\|_{\mathcal{X}^1} \leq \|\theta\|_{\mathcal{X}^1}$ and $\|u_\theta\|_{\mathcal{X}^{1-2\alpha}} \leq \|\theta\|_{\mathcal{X}^{1-2\alpha}}$, we get

$$\begin{aligned} \left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} \operatorname{div}(\theta u_\theta) dz \right\|_{\mathcal{X}^{1-2\alpha}} &\leq 4 \int_0^t \|\theta\|_{\mathcal{X}^{1-2\alpha}} \|\theta\|_{\mathcal{X}^1} dz \\ &\leq 4 \|\theta\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})} \|\theta\|_{L_T^1(\mathcal{X}^1)}. \end{aligned}$$

Also the following lemma will be useful in the sequel.

Lemma 2.4. *Let $\theta \in L_T^\infty(\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)) \cap L_T^1(\mathcal{X}^1(\mathbb{R}^2))$. Then*

$$\int_0^T \left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} \operatorname{div}(\theta u_\theta) dz \right\|_{\mathcal{X}^1} dt \leq 4 \|\theta\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})} \|\theta\|_{L_T^1(\mathcal{X}^1)}.$$

Proof. We have

$$\begin{aligned} \int_0^T \left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} \operatorname{div}(\theta u_\theta) dz \right\|_{\mathcal{X}^1} dt &\leq \int_0^T \int_0^t \|e^{-(t-z)|D|^{2\alpha}} \operatorname{div}(\theta u_\theta)(z)\|_{\mathcal{X}^1} dz dt \\ &\leq \int_0^T \int_0^t \int_\xi e^{-(t-z)|\xi|^{2\alpha}} |\xi|^2 |\mathcal{F}(\theta u_\theta)(z, \xi)| d\xi dz dt \\ &\leq \int_\xi |\xi|^2 \left(\int_0^T \int_0^t e^{-(t-z)|\xi|^{2\alpha}} |\mathcal{F}(\theta u_\theta)(z, \xi)| dz dt \right) d\xi. \end{aligned}$$

By integrating the function $e^{-(t-z)|\xi|^{2\alpha}}$ twice with respect to $z \in [0, t]$ and with respect to $t \in [0, T]$, we get

$$\int_0^T \int_0^t e^{-(t-z)|\xi|^{2\alpha}} |\mathcal{F}(\theta u_\theta)(z, \xi)| dz dt = \int_0^T |\mathcal{F}(\theta u)(z, \xi)| \left(\int_z^T e^{-(t-z)|\xi|^{2\alpha}} dt \right) dz.$$

Then

$$\begin{aligned}
\int_0^T \left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} \operatorname{div}(\theta u_\theta) dz \right\|_{\mathcal{X}^1} dt &\leq \int_\xi |\xi|^2 \left(\int_0^T \left[\int_z^T e^{-(t-z)|\xi|^{2\alpha}} dt \right] |\mathcal{F}(\theta u_\theta)(z, \xi)| dz \right) d\xi \\
&\leq \int_\xi |\xi|^2 \left(\int_0^T \left[\frac{1 - e^{-(T-z)|\xi|^{2\alpha}}}{|\xi|^{2\alpha}} \right] |\mathcal{F}(\theta u_\theta)(z, \xi)| dz \right) d\xi \\
&\leq \int_\xi |\xi|^{2-2\alpha} \int_0^T |\mathcal{F}(\theta u_\theta)(z, \xi)| dz d\xi \\
&\leq \int_0^T \|(\theta u_\theta)(z)\|_{\mathcal{X}^{2-2\alpha}} dz.
\end{aligned}$$

Using Lemma 2.2 and the fact $\|u_\theta\|_{\mathcal{X}^1} \leq \|\theta\|_{\mathcal{X}^1}$ and $\|u_\theta\|_{\mathcal{X}^{1-2\alpha}} \leq \|\theta\|_{\mathcal{X}^{1-2\alpha}}$, we get

$$\begin{aligned}
\int_0^T \left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} \operatorname{div}(\theta u_\theta) dz \right\|_{\mathcal{X}^1} dt &\leq 4 \int_0^T \|\theta(z)\|_{\mathcal{X}^{2-2\alpha}} \|\theta(z)\|_{\mathcal{X}^1} dz \\
&\leq 4 \|\theta\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})} \|\theta\|_{L_T^1(\mathcal{X}^1)}.
\end{aligned}$$

3. PROOF OF THEOREM 1.1

3.1. Existence. The idea of the proof is to write the initial condition as sum higher and lower frequencies. For small frequencies, we give a regular solution of the associated linear system to (\mathbf{S}_α) and for the higher frequencies we consider a partial differential equation very similar to (\mathbf{S}_α) with small initial data in $\mathcal{X}^{1-2\alpha}$ for which we can solve it by the fixed point theorem.

- Let r be a real number such that $0 < r < 1/20$.
- Let $N \in \mathbb{N}$, such that

$$\int_{|\xi| > N} |\xi|^{1-2\alpha} |\widehat{\theta^0}(\xi)| d\xi < \frac{r}{5}.$$

Put a^0 and b^0 defined by

$$\begin{aligned}
a^0 &= \mathcal{F}^{-1}(\mathbf{1}_{|\xi| < N} \widehat{\theta^0}(\xi)) \\
b^0 &= \mathcal{F}^{-1}(\mathbf{1}_{|\xi| > N} \widehat{\theta^0}(\xi)).
\end{aligned}$$

Clearly

$$(3.1) \quad \|b^0\|_{\mathcal{X}^{1-2\alpha}} < \frac{r}{5}.$$

And put

$$a = e^{-t|D|^{2\alpha}} a^0,$$

a is the unique solution of the heat equation

$$\begin{cases} \partial_t a + (-\Delta)^\alpha a &= 0 \\ a(0) &= a^0. \end{cases}$$

We have

$$\|a\|_{\mathcal{X}^{1-2\alpha}} \leq \|\theta^0\|_{\mathcal{X}^{1-2\alpha}}, \quad \forall t \geq 0,$$

and

$$\begin{aligned}
\|a\|_{L^1([0,T],\mathcal{X}^1)} &= \int_0^T \int_{\xi} e^{-t|\xi|^{2\alpha}} |\xi| |\widehat{\theta^0}(\xi)| d\xi dt \\
&= \int_{\xi} \left(\int_0^T e^{-t|\xi|^{2\alpha}} dt \right) |\xi| |\widehat{\theta^0}(\xi)| d\xi \\
&= \int_{\xi} (1 - e^{-T|\xi|^{2\alpha}}) |\xi|^{1-2\alpha} |\widehat{\theta^0}(\xi)| d\xi.
\end{aligned}$$

Using Dominated Convergence Theorem, we get

$$(3.2) \quad \lim_{T \rightarrow 0^+} \|a\|_{L^1([0,T],\mathcal{X}^1)} = 0.$$

Let $\varepsilon > 0$ such that

$$(3.3) \quad 4\varepsilon \|\theta^0\|_{\mathcal{X}^{1-2\alpha}} < \frac{r}{5}$$

and

$$(3.4) \quad 2 \left(\|\theta^0\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}} \varepsilon^{\frac{1}{2\alpha}} + \|\theta^0\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}} \varepsilon^{1-\frac{1}{2\alpha}} \right) < \frac{1}{5}.$$

By equation (3.2), there is a time $T = T(\varepsilon) > 0$ such that

$$(3.5) \quad \|a\|_{L^1([0,T],\mathcal{X}^1)} < \varepsilon.$$

Put $b = \theta - a$, clearly b is a solution of the following system

$$\begin{cases} \partial_t b + (-\Delta)^{\alpha} b + (u_a + u_b) \nabla(a + b) = 0 \\ b(0) = b^0. \end{cases}$$

The integral form of b is as follow

$$b = e^{-t|D|^{2\alpha}} b^0 - \int_0^t e^{-(t-z)|D|^{2\alpha}} (u_a + u_b) \nabla(a + b) dz.$$

To prove the existence of b , put the following operator

$$\Psi(b) = e^{-t|D|^{2\alpha}} b^0 - \int_0^t e^{-(t-z)|D|^{2\alpha}} (u_a + u_b) \nabla(a + b) dz.$$

- Now, we introduce the space \mathcal{X}_T as follows

$$\mathcal{X}_T = \mathcal{C}([0, T], \mathcal{X}^{1-2\alpha}(\mathbb{R}^2)) \cap L^1([0, T], \mathcal{X}^1(\mathbb{R}^2))$$

with the norm

$$\|f\|_{\mathcal{X}_T} = \|f\|_{L_T^{\infty}(\mathcal{X}^{1-2\alpha})} + \|f\|_{L_T^1(\mathcal{X}^1)}.$$

Using Lemmas 2.3 and 2.4, we can prove $\Psi(\mathcal{X}_T) \subset \mathcal{X}_T$.

- Also, we note \mathbf{B}_r the subset of \mathcal{X}_T defined by

$$\mathbf{B}_r = \{\theta \in \mathcal{X}_T; \|\theta\|_{L^{\infty}([0,T],\mathcal{X}^{1-2\alpha})} \leq r, \|\theta\|_{L^1([0,T],\mathcal{X}^1)} \leq r\}.$$

- Let $b \in \mathbf{B}_r$. Prove that $\Psi(b) \in \mathbf{B}_r$. We have

$$\|\Psi(b)(t)\|_{\mathcal{X}^{1-2\alpha}} \leq \sum_{k=0}^4 I_k,$$

where

$$\begin{aligned}
I_0 &= \|e^{-t|D|^{2\alpha}} b^0\|_{\mathcal{X}^{1-2\alpha}} \\
I_1 &= \int_0^t \|e^{-(t-z)|D|^{2\alpha}} u_a \nabla a\|_{\mathcal{X}^{1-2\alpha}} dz \\
I_2 &= \int_0^t \|e^{-(t-z)|D|^{2\alpha}} u_a \nabla b\|_{\mathcal{X}^{1-2\alpha}} dz \\
I_3 &= \int_0^t \|e^{-(t-z)|D|^{2\alpha}} u_b \nabla a\|_{\mathcal{X}^{1-2\alpha}} dz \\
I_4 &= \int_0^t \|e^{-(t-z)|D|^{2\alpha}} u_b \nabla b\|_{\mathcal{X}^{1-2\alpha}} dz.
\end{aligned}$$

Using equation (3.1), Lemma 2.3 and the fact $b \in \mathbf{B}_r$, we get

$$\begin{aligned}
I_0 &\leq \frac{r}{5} \\
I_1 &\leq 4\|a\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}\|a\|_{L_T^1(\mathcal{X}^1)} \leq 4\varepsilon\|\theta^0\|_{(\mathcal{X}^{1-2\alpha})} \leq \frac{r}{5} \\
I_2, I_3 &\leq 2\|a\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{1-\frac{1}{2\alpha}}\|a\|_{L_T^1(\mathcal{X}^1)}^{\frac{1}{2\alpha}}\|b\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{\frac{1}{2\alpha}}\|b\|_{L_T^1(\mathcal{X}^1)}^{1-\frac{1}{2\alpha}} \\
&\quad + 2\|a\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{\frac{1}{2\alpha}}\|a\|_{L_T^1(\mathcal{X}^1)}^{1-\frac{1}{2\alpha}}\|b\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{1-\frac{1}{2\alpha}}\|b\|_{L_T^1(\mathcal{X}^1)}^{\frac{1}{2\alpha}} \\
&\leq 2\left(\|\theta^0\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}}\varepsilon^{\frac{1}{2\alpha}} + \|\theta^0\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}}\varepsilon^{1-\frac{1}{2\alpha}}\right)r \\
&\leq \frac{r}{5} \\
I_4 &\leq 4\|b\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}\|b\|_{L_T^1(\mathcal{X}^1)} \leq 4r^2 \leq \frac{r}{5}.
\end{aligned}$$

Then

$$(3.6) \quad \|\Psi(b)(t)\|_{\mathcal{X}^{1-2\alpha}} \leq r.$$

Similarly,

$$\|\Psi(b)(t)\|_{L_T^1(\mathcal{X}^1)} \leq \sum_{k=0}^4 J_k,$$

where

$$\begin{aligned}
J_0 &= \int_0^T \|e^{-t|D|^{2\alpha}} b^0\|_{\mathcal{X}^1} dt, \\
J_1 &= \int_0^T \left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} u_a \nabla a dz \right\|_{\mathcal{X}^1} dt, \\
J_2 &= \int_0^T \left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} u_a \nabla b dz \right\|_{\mathcal{X}^1} dt, \\
J_3 &= \int_0^T \left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} u_b \nabla a dz \right\|_{\mathcal{X}^1} dt, \\
J_4 &= \int_0^T \left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} u_b \nabla b dz \right\|_{\mathcal{X}^1} dt.
\end{aligned}$$

By using Lemma 2.4, we have

$$\begin{aligned}
J_0 &\leq \frac{r}{5} \\
J_1 &\leq 4\|a\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}\|a\|_{L_T^1(\mathcal{X}^1)} \leq 4\varepsilon\|\theta^0\|_{(\mathcal{X}^{1-2\alpha})} \leq \frac{r}{5} \\
J_2, J_3 &\leq 2\|a\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{1-\frac{1}{2\alpha}}\|a\|_{L_T^1(\mathcal{X}^1)}^{\frac{1}{2\alpha}}\|b\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{1-\frac{1}{2\alpha}}\|b\|_{L_T^1(\mathcal{X}^1)}^{\frac{1}{2\alpha}} \\
&\quad + 2\|a\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{\frac{1}{2\alpha}}\|a\|_{L_T^1(\mathcal{X}^1)}^{1-\frac{1}{2\alpha}}\|b\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{1-\frac{1}{2\alpha}}\|b\|_{L_T^1(\mathcal{X}^1)}^{\frac{1}{2\alpha}} \\
&\leq 2\left(\|\theta^0\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}}\varepsilon^{\frac{1}{2\alpha}} + \|\theta^0\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}}\varepsilon^{1-\frac{1}{2\alpha}}\right)r \\
&\leq \frac{r}{5} \\
J_4 &\leq 4\|b\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}\|b\|_{L_T^1(\mathcal{X}^1)} \leq 4r^2 \leq \frac{r}{5}.
\end{aligned}$$

Then

$$(3.7) \quad \|\Psi(b)(t)\|_{L_T^1(\mathcal{X}^1)} \leq r.$$

Combining equation (3.6) and (3.7), we get $\Psi(b) \in \mathbf{B}_r$ and we can deduce

$$(3.8) \quad \Psi(\mathbf{B}_r) \subset \mathbf{B}_r.$$

• Prove that

$$\|\Psi(b_1) - \Psi(b_2)\|_{\mathcal{X}_T} \leq \frac{1}{2}\|b_1 - b_2\|_{\mathcal{X}_T}, \quad b_1, b_2 \in \mathbf{B}_r.$$

Put B_1 and B_2 defined by

$$B_i = (\partial_{x_2}\Lambda^{-1}b_i, -\partial_{x_1}\Lambda^{-1}b_i).$$

We have

$$\begin{aligned}
\Psi(b_1) - \Psi(b_2) &= - \int_0^t e^{-(t-z)|D|^{2\alpha}} (B_2 \nabla b_2 - B_1 \nabla b_1) \\
&= - \int_0^t e^{-(t-z)|D|^{2\alpha}} (B_2 \nabla(b_2 - b_1) + (B_2 - B_1) \nabla b_1)
\end{aligned}$$

and

$$\|\Psi(b_1) - \Psi(b_2)\|_{\mathcal{X}^{1-2\alpha}} \leq K_1 + K_2,$$

with

$$\begin{aligned}
K_1 &= \left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} B_2 \nabla(b_2 - b_1) dz \right\|_{\mathcal{X}^{1-2\alpha}}, \\
K_2 &= \left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} (B_2 - B_1) \nabla b_1 dz \right\|_{\mathcal{X}^{1-2\alpha}}.
\end{aligned}$$

Using Lemmas 2.2 and 2.3, we can deduce

$$\begin{aligned}
K_1 &\leq 2\|B_2\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{\frac{1}{2\alpha}}\|B_2\|_{L_T^1(\mathcal{X}^1)}^{1-\frac{1}{2\alpha}}\|b_2 - b_1\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{1-\frac{1}{2\alpha}}\|b_2 - b_1\|_{L_T^1(\mathcal{X}^1)}^{\frac{1}{2\alpha}} \\
&\quad + 2\|B_2\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{1-\frac{1}{2\alpha}}\|B_2\|_{L_T^1(\mathcal{X}^1)}^{\frac{1}{2\alpha}}\|b_2 - b_1\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{\frac{1}{2\alpha}}\|b_2 - b_1\|_{L_T^1(\mathcal{X}^1)}^{1-\frac{1}{2\alpha}} \\
&\leq 2\|b_2\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{\frac{1}{2\alpha}}\|b_2\|_{L_T^1(\mathcal{X}^1)}^{1-\frac{1}{2\alpha}}\|b_2 - b_1\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{1-\frac{1}{2\alpha}}\|b_2 - b_1\|_{L_T^1(\mathcal{X}^1)}^{\frac{1}{2\alpha}} \\
&\quad + 2\|b_2\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{1-\frac{1}{2\alpha}}\|b_2\|_{L_T^1(\mathcal{X}^1)}^{\frac{1}{2\alpha}}\|b_2 - b_1\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{\frac{1}{2\alpha}}\|b_2 - b_1\|_{L_T^1(\mathcal{X}^1)}^{1-\frac{1}{2\alpha}}.
\end{aligned}$$

Then

$$\begin{aligned} K_1 &\leq 2r \|b_2 - b_1\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{1-\frac{1}{2\alpha}} \|b_2 - b_1\|_{L_T^1(\mathcal{X}^1)}^{\frac{1}{2\alpha}} \\ &\quad + 2r \|b_2 - b_1\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}^{\frac{1}{2\alpha}} \|b_2 - b_1\|_{L_T^1(\mathcal{X}^1)}^{1-\frac{1}{2\alpha}} \\ &\leq 4r \|b_2 - b_1\|_{\mathcal{X}_T}. \end{aligned}$$

Similarly, we get

$$K_2 \leq 4r \|b_2 - b_1\|_{\mathcal{X}_T}.$$

Using the fact $r < 1/20$, we obtain

$$(3.9) \quad \|\Psi(b_2) - \Psi(b_1)\|_{\mathcal{X}_T} \leq \frac{1}{2} \|b_2 - b_1\|_{\mathcal{X}_T}, \quad \forall b_1, b_2 \in \mathbf{B}_r.$$

Then, combining equations (3.8)-(3.9) and the fixed point theorem, there is a unique $b \in \mathbf{B}_r$ such that $\theta = a + b$ is solution of (\mathbf{S}_α) with $\theta \in \mathcal{X}_T(\mathbb{R}^2)$.

3.2. Uniqueness. Let $\theta_1, \theta_2 \in \mathcal{C}([0, T], \mathcal{X}^{1-2\alpha}(\mathbb{R}^2))$ be two solution of (\mathbf{S}_α) , with $\theta_1 \in L^1([0, T], \mathcal{X}^{1-2\alpha}(\mathbb{R}^2))$ and $\theta_1(0) = \theta_2(0)$. Put u_1, u_2 , δ and w defined as follows

$$\begin{aligned} u_i &= (\partial_{x_2} \Lambda^{-1} \theta_i, -\partial_{x_1} \Lambda^{-1} \theta_i). \\ \delta &= \theta_1 - \theta_2 \\ w &= u_1 - u_2 \end{aligned}$$

We have

$$(3.10) \quad \partial_t \delta + (-\Delta)^\alpha \delta + w \nabla \delta + w \nabla \theta_1 + u_1 \nabla \delta = 0$$

then

$$\partial_t \widehat{\delta} + |\xi|^{2\alpha} \widehat{\delta} + \mathcal{F}(w \nabla \delta) + \mathcal{F}(w \nabla \theta_1) + \mathcal{F}(u_1 \nabla \delta) = 0$$

multiply this equation by $\widehat{\delta}$, we get

$$(3.11) \quad \partial_t \widehat{\delta} \cdot \widehat{\delta} + |\xi|^{2\alpha} |\widehat{\delta}|^2 + \mathcal{F}(w \nabla \delta) \cdot \widehat{\delta} + \mathcal{F}(w \nabla \theta_1) \cdot \widehat{\delta} + \mathcal{F}(u_1 \nabla \delta) \cdot \widehat{\delta} = 0.$$

From equation (3.10), we have

$$\partial_t \widehat{\delta} + |\xi|^{2\alpha} \widehat{\delta} + \overline{\mathcal{F}(w \nabla \delta)} + \overline{\mathcal{F}(w \nabla \theta_1)} + \overline{\mathcal{F}(u_1 \nabla \delta)} = 0$$

multiply this equation by $\widehat{\delta}$, we get

$$(3.12) \quad \partial_t \widehat{\delta} \cdot \widehat{\delta} + |\xi|^{2\alpha} |\widehat{\delta}|^2 + \overline{\mathcal{F}(w \nabla \delta)} \cdot \widehat{\delta} + \overline{\mathcal{F}(w \nabla \theta_1)} \cdot \widehat{\delta} + \overline{\mathcal{F}(u_1 \nabla \delta)} \cdot \widehat{\delta} = 0.$$

By summing (3.11) and (3.12), we get

$$\partial_t |\widehat{\delta}|^2 + 2|\xi|^{2\alpha} |\widehat{\delta}|^2 + 2\operatorname{Re}(\mathcal{F}(w \nabla \delta) \cdot \widehat{\delta}) + 2\operatorname{Re}(\mathcal{F}(w \nabla \theta_1) \cdot \widehat{\delta}) + 2\operatorname{Re}(\mathcal{F}(u_1 \nabla \delta) \cdot \widehat{\delta}) = 0$$

and

$$\partial_t |\widehat{\delta}|^2 + 2|\xi|^{2\alpha} |\widehat{\delta}|^2 \leq 2|\mathcal{F}(w \nabla \delta)| \cdot |\widehat{\delta}| + 2|\mathcal{F}(w \nabla \theta_1)| \cdot |\widehat{\delta}| + 2|\mathcal{F}(u_1 \nabla \delta)| \cdot |\widehat{\delta}|.$$

For $\varepsilon > 0$, we have

$$\partial_t |\widehat{\delta}|^2 = \partial_t (|\widehat{\delta}|^2 + \varepsilon^2) = 2\sqrt{|\widehat{\delta}|^2 + \varepsilon^2} \partial_t \sqrt{|\widehat{\delta}|^2 + \varepsilon^2}$$

then

$$2\partial_t \sqrt{|\widehat{\delta}|^2 + \varepsilon^2} + 2|\xi|^{2\alpha} \frac{|\widehat{\delta}|^2}{\sqrt{|\widehat{\delta}|^2 + \varepsilon^2}} \leq 2|\mathcal{F}(w \nabla \delta)| \frac{|\widehat{\delta}|}{\sqrt{|\widehat{\delta}|^2 + \varepsilon^2}} + 2|\mathcal{F}(w \nabla \theta_1)| \frac{|\widehat{\delta}|}{\sqrt{|\widehat{\delta}|^2 + \varepsilon^2}} + 2|\mathcal{F}(u_1 \nabla \delta)| \frac{|\widehat{\delta}|}{\sqrt{|\widehat{\delta}|^2 + \varepsilon^2}}$$

$$\begin{aligned}
& + 2|\mathcal{F}(u_1 \nabla \delta)| \frac{|\widehat{\delta}|}{\sqrt{|\widehat{\delta}|^2 + \varepsilon^2}} \\
& \leq 2|\mathcal{F}(w \nabla \delta)| + 2|\mathcal{F}(w \nabla \theta_1)| + 2|\mathcal{F}(u_1 \nabla \delta)|.
\end{aligned}$$

By integrating with respect to time

$$\sqrt{|\widehat{\delta}|^2 + \varepsilon^2} + \int_0^t |\xi|^{2\alpha} \frac{|\widehat{\delta}|^2}{\sqrt{|\widehat{\delta}|^2 + \varepsilon^2}} \leq \int_0^t |\mathcal{F}(w \nabla \delta)| + \int_0^t |\mathcal{F}(w \nabla \theta_1)| + \int_0^t |\mathcal{F}(u_1 \nabla \delta)|.$$

Letting $\varepsilon \rightarrow 0$, we get

$$|\widehat{\delta}| + \int_0^t |\xi|^{2\alpha} |\widehat{\delta}| \leq \int_0^t |\mathcal{F}(w \nabla \delta)| + \int_0^t |\mathcal{F}(w \nabla \theta_1)| + \int_0^t |\mathcal{F}(u_1 \nabla \delta)|.$$

Multiplying by $|\xi|^{1-2\alpha}$ and integrating with respect to ξ , we get

$$\begin{aligned}
\|\delta\|_{\mathcal{X}^{1-2\alpha}} + \int_0^t \|\delta\|_{\mathcal{X}^1} & \leq \int_0^t \|w \nabla \delta\|_{\mathcal{X}^{1-2\alpha}} + \int_0^t \|w \nabla \theta_1\|_{\mathcal{X}^{1-2\alpha}} + \int_0^t \|u_1 \nabla \delta\|_{\mathcal{X}^{1-2\alpha}}. \\
& \leq \int_0^t \|\delta w\|_{\mathcal{X}^{2-2\alpha}} + \int_0^t \|\theta_1 w\|_{\mathcal{X}^{2-2\alpha}} + \int_0^t \|\delta u_1\|_{\mathcal{X}^{2-2\alpha}}.
\end{aligned}$$

By Lemma 2.2, we get

$$\begin{aligned}
\|\delta\|_{\mathcal{X}^{1-2\alpha}} + \int_0^t \|\delta\|_{\mathcal{X}^1} & \leq 4 \int_0^t \|\delta\|_{\mathcal{X}^{1-2\alpha}} \|\delta\|_{\mathcal{X}^1} + 2 \int_0^t \|\delta\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}} \|\delta\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}} \|\theta_1\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}} \|\theta_1\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}} \\
& + 2 \int_0^t \|\delta\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}} \|\delta\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}} \|\theta_1\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}} \|\theta_1\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}}.
\end{aligned}$$

Using the elementary inequality

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$

with

$$p = \frac{1}{2\alpha}, \quad q = \frac{2\alpha}{2\alpha-1},$$

we get

$$\|\delta\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}} \|\delta\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}} \|\theta_1\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}} \|\theta_1\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}} \leq \frac{1}{4} \|\delta\|_{\mathcal{X}^1} + c_\alpha \|\delta\|_{\mathcal{X}^{1-2\alpha}} \|\theta_1\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha-1}} \|\theta_1\|_{\mathcal{X}^1}$$

and

$$\|\delta\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}} \|\delta\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}} \|\theta_1\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}} \|\theta_1\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}} \leq \frac{1}{4} \|\delta\|_{\mathcal{X}^1} + c'_\alpha \|\delta\|_{\mathcal{X}^{1-2\alpha}} \|\theta_1\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha-1}} \|\theta_1\|_{\mathcal{X}^1}.$$

Then

$$\|\delta\|_{\mathcal{X}^{1-2\alpha}} + \int_0^t \|\delta\|_{\mathcal{X}^1} \leq (c_\alpha + c'_\alpha) \int_0^t \|\delta\|_{\mathcal{X}^{1-2\alpha}} \|\theta_1\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha-1}} \|\theta_1\|_{\mathcal{X}^1}.$$

Using Gronwall Lemma and the fact $(t \mapsto \|\theta_1\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha-1}} \|\theta_1\|_{\mathcal{X}^1}) \in L^1([0, T])$, we can deduce that $\delta = 0$ in $[0, T]$ which gives the uniqueness.

3.3. Small initial data. In this section, we suppose that $\|\theta^0\|_{\mathcal{X}^{1-2\alpha}} < 1/4$.

Pass to Fourier transform of the first equation of (\mathbf{S}_α) and multiply it by $\bar{\bar{\theta}}$, we get

$$(3.13) \quad \partial_t \widehat{\bar{\theta}} + |\xi|^{2\alpha} \widehat{\bar{\theta}} + \overline{\mathcal{F}(u \nabla \theta)} \cdot \widehat{\bar{\theta}} = 0.$$

Similarly, we have

$$\partial_t \bar{\bar{\theta}} + |\xi|^{2\alpha} \bar{\bar{\theta}} + \overline{\mathcal{F}(u \nabla \theta)} = 0$$

and

$$(3.14) \quad \partial_t \bar{\bar{\theta}} \cdot \widehat{\theta} + |\xi|^{2\alpha} \widehat{\theta} + \overline{\mathcal{F}(u \nabla \theta)} \cdot \widehat{\theta} = 0.$$

By summing (3.13) and (3.14), we get

$$\partial_t |\widehat{\theta}|^2 + 2|\xi|^{2\alpha} |\widehat{\theta}|^2 + 2\operatorname{Re}(\mathcal{F}(u \nabla \theta) \cdot \widehat{\bar{\theta}}) = 0$$

and

$$\partial_t |\widehat{\theta}|^2 + 2|\xi|^{2\alpha} |\widehat{\theta}|^2 \leq 2|\mathcal{F}(u \nabla \theta)| \cdot |\widehat{\bar{\theta}}|.$$

For $\varepsilon > 0$, we have

$$\partial_t |\widehat{\theta}|^2 = \partial_t (|\widehat{\theta}|^2 + \varepsilon^2) = 2\sqrt{|\widehat{\theta}|^2 + \varepsilon^2} \partial_t \sqrt{|\widehat{\theta}|^2 + \varepsilon^2}.$$

Then

$$\begin{aligned} 2\partial_t \sqrt{|\widehat{\theta}|^2 + \varepsilon^2} + 2|\xi|^{2\alpha} \frac{|\widehat{\theta}|^2}{\sqrt{|\widehat{\theta}|^2 + \varepsilon^2}} &\leq 2|\mathcal{F}(u \nabla \theta)| \cdot \frac{|\widehat{\theta}|}{\sqrt{|\widehat{\theta}|^2 + \varepsilon^2}} \\ &\leq 2|\mathcal{F}(u \nabla \theta)|. \end{aligned}$$

By integrating with respect to time

$$\sqrt{|\widehat{\theta}|^2 + \varepsilon^2} + \int_0^t |\xi|^{2\alpha} \frac{|\widehat{\theta}|^2}{\sqrt{|\widehat{\theta}|^2 + \varepsilon^2}} \leq \sqrt{|\widehat{\theta}^0|^2 + \varepsilon^2} + \int_0^t |\mathcal{F}(u \nabla \theta)|.$$

Letting $\varepsilon \rightarrow 0$, we get

$$|\widehat{\theta}| + \int_0^t |\xi|^{2\alpha} |\widehat{\theta}| \leq |\widehat{\theta}^0| + \int_0^t |\mathcal{F}(u \nabla \theta)|.$$

Multiplying by $|\xi|^{1-2\alpha}$ and integrating with respect to ξ , we get

$$\|\theta\|_{\mathcal{X}^{1-2\alpha}} + \int_0^t \|\theta\|_{\mathcal{X}^1} \leq \|\theta^0\|_{\mathcal{X}^{1-2\alpha}} + \int_0^t \|u \nabla \theta\|_{\mathcal{X}^{1-2\alpha}}.$$

By lemma 2.2, we get

$$\begin{aligned} \|\theta\|_{\mathcal{X}^{1-2\alpha}} + \int_0^t \|\theta\|_{\mathcal{X}^1} &\leq \|\theta^0\|_{\mathcal{X}^{1-2\alpha}} + 4 \int_0^t \|\theta\|_{\mathcal{X}^{1-2\alpha}} \|\theta\|_{\mathcal{X}^1} \\ &\leq \|\theta^0\|_{\mathcal{X}^{1-2\alpha}} + 4 \sup_{z \in [0, t]} \|\theta(z)\|_{\mathcal{X}^{1-2\alpha}} \int_0^t \|\theta\|_{\mathcal{X}^1}. \end{aligned}$$

Let $T = \sup\{t > 0; \sup_{z \in [0, t]} \|\theta(z)\|_{\mathcal{X}^{1-2\alpha}} < \frac{1}{4}\}$. By the above equation, we have

$$\|\theta(t)\|_{\mathcal{X}^{1-2\alpha}} \leq \|\theta^0\|_{\mathcal{X}^{1-2\alpha}} < \frac{1}{4}, \forall t \in [0, T),$$

then $T = \infty$. Therefore the global existence and inequality (1.2) are proved.

4. PROOF OF THEOREM 1.2

The same approach of Theorem 1.1 is used to obtain a blow up result of $\theta \in \mathcal{C}([0, T^*), \mathcal{X}^{1-2\alpha}(\mathbb{R}^2))$ if $T^* < \infty$.

Assume that $\int_0^{T^*} \|\theta(t)\|_{\mathcal{X}^1} dt < \infty$. Let $0 < T_0 < T^*$ such that

$$\int_{T_0}^{T^*} \|\theta(t)\|_{\mathcal{X}^1} dt < \frac{1}{2}.$$

For $t \in [T_0, T^*)$ and $s \in [T_0, t]$

$$\begin{aligned} \|\theta(s)\|_{\mathcal{X}^{1-2\alpha}} + \int_{T_0}^s \|\theta(t)\|_{\mathcal{X}^1} dt &\leq \|\theta(T_0)\|_{\mathcal{X}^{1-2\alpha}} + \int_{T_0}^s \|\theta(t)\|_{\mathcal{X}^{1-2\alpha}} \|\theta(t)\|_{\mathcal{X}^1} dt \\ &\leq \|\theta(T_0)\|_{\mathcal{X}^{1-2\alpha}} + \sup_{T_0 \leq z \leq t} \|\theta(z)\|_{\mathcal{X}^{1-2\alpha}} \int_{T_0}^t \|\theta(t)\|_{\mathcal{X}^1} dt \\ &\leq \|\theta(T_0)\|_{\mathcal{X}^{1-2\alpha}} + \frac{1}{2} \sup_{T_0 \leq z \leq t} \|\theta(z)\|_{\mathcal{X}^{1-2\alpha}} \end{aligned}$$

then

$$\sup_{0 \leq s \leq t} \|\theta(s)\|_{\mathcal{X}^{1-2\alpha}} \leq \|\theta(T_0)\|_{\mathcal{X}^{1-2\alpha}} + \frac{1}{2} \sup_{T_0 \leq z \leq t} \|\theta(z)\|_{\mathcal{X}^{1-2\alpha}}.$$

We can deduce that

$$\sup_{0 \leq s \leq t} \|\theta(s)\|_{\mathcal{X}^{1-2\alpha}} \leq 2\|\theta(T_0)\|_{\mathcal{X}^{1-2\alpha}}, \quad \forall t \in [T_0, T^*).$$

Put

$$M = \max \left(2\|\theta(T_0)\|_{\mathcal{X}^{1-2\alpha}}; \max_{t \in [0, T_0]} \|\theta(t)\|_{\mathcal{X}^{1-2\alpha}} \right).$$

We have

$$\|\theta(t)\|_{\mathcal{X}^{1-2\alpha}} \leq M, \quad \forall t \in [0, T^*).$$

Using the integral form of θ , we can write, for $t < t' \in [0, T^*)$,

$$\theta(t') - \theta(t) = L_1(t, t') + L_2(t, t')$$

with

$$\begin{aligned} L_1 &= \int_0^t \left(1 - e^{-(t'-t)|D|^{2\alpha}} \right) e^{-(t-z)|D|^{2\alpha}} (u \cdot \nabla \theta)(z) dz, \\ L_2 &= \int_t^{t'} e^{-(t'-z)|D|^{2\alpha}} (u \cdot \nabla \theta)(z) dz. \end{aligned}$$

We have

$$\begin{aligned} \|L_1(t, t')\|_{\mathcal{X}^{1-2\alpha}} &\leq \int_0^t \int_{\xi} \left(1 - e^{-(t'-t)|\xi|^{2\alpha}} \right) e^{-(t-z)|\xi|^{2\alpha}} |\xi|^{1-2\alpha} |\mathcal{F}(u \cdot \nabla \theta)(z, \xi)| d\xi dz \\ &\leq \int_0^t \int_{\xi} \left(1 - e^{-(t'-t)|\xi|^{2\alpha}} \right) |\xi|^{1-2\alpha} |\mathcal{F}(u \cdot \nabla \theta)(z, \xi)| d\xi dz. \\ \|L_2(t, t')\|_{\mathcal{X}^{1-2\alpha}} &\leq \int_t^{t'} \int_{\xi} e^{-(t'-z)|\xi|^{2\alpha}} |\xi|^{1-2\alpha} |\mathcal{F}(u \cdot \nabla \theta)(z, \xi)| d\xi dz \\ &\leq \int_t^{t'} \int_{\xi} |\xi|^{1-2\alpha} |\mathcal{F}(u \cdot \nabla \theta)(z, \xi)| d\xi dz. \end{aligned}$$

Using Lemmas 2.2-2.3 and Dominated Convergence Theorem, we can deduce that

$$\limsup_{\substack{t, t' \nearrow T^* \\ t < t'}} \|L_1(t, t')\|_{\mathcal{X}^{1-2\alpha}} = 0, \quad \limsup_{\substack{t, t' \nearrow T^* \\ t < t'}} \|L_2(t, t')\|_{\mathcal{X}^{1-2\alpha}} = 0.$$

Therefore

$$\limsup_{\substack{t, t' \nearrow T^* \\ t < t'}} \|\theta(t') - \theta(t)\|_{\mathcal{X}^{1-2\alpha}} = 0.$$

Then $\theta(t)$ is of Cauchy type at the left of T^* in the Banach space $\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$. Then, there is θ^* an element of $\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$ such that

$$\lim_{t \nearrow T^*} \theta(t) = \theta^*.$$

Now, consider the following system

$$\begin{cases} \partial_t a + (-\Delta)^\alpha a + u_a \cdot \nabla a = 0 \\ a(0) = \theta^* \end{cases}$$

By Theorem 1.1, there a time $t_0 > 0$ and unique solution a such that $a \in \mathcal{C}([0, t_0], \mathcal{X}^{1-2\alpha}(\mathbb{R}^2))$. Then

$$\Theta(t) = \begin{cases} \theta(t), & \text{if } t \in [0, T^*) \\ a(t - T^*), & \text{if } t \in [T^*, T^* + t_0] \end{cases}$$

is a solution of (\mathbf{S}_α) with initial data θ^0 on the interval $[0, T^* + t_0]$ which contradicts the maximality of T^* .

REFERENCES

- [1] H. Abidi, T. Hmidi, *On the global well-posedness of the critical quasi-geostrophic equation*, SIAM J. Math. Anal. 40 (1) (2008) 167185.
- [2] T. Hmidi, S. Keraani, *Global solutions of the super-critical 2D quasi-geostrophic equation in Besov spaces*, Adv. Math. 214 (2) (2007) 618638.
- [3] M. Cannone, *Harmonic analysis tools for solving the incompressible Navier-Stokes equations*, Diterot Editeur, Paris, 1995.
- [4] H. Bae, *Global well-posedness of dissipative quasi-geostrophic equations in the critical spaces*, Proc. Amer. Math. Soc. 136 (2008) 257262.
- [5] H. Dong, D. Du, *Global well-posedness and a decay estimate for the critical dissipative quasi-geostrophic equation in the whole space*, Discrete Contin. Dyn. Syst. 21 (4) (2008) 10951101.
- [6] Z. Lei and F. Lin, *Global Mild Solutions of Navier-Stokes Equations*, Communications on Pure and Applied Mathematics, Vol. LXIV, (2011) pages 1297-1304.
- [7] H. Fujita, T. Kato, *On the NavierStokes initial value problem, I*, Arch. Ration. Mech. Anal. 16 (1964) 269315.
- [8] Z. Zhang and Z. Yin, *Global Well-posedness for the Generalized Navier-Stokes System*, arXiv: 1306.3735v1 [math.Ap] 17 June 2013.
- [9] J.A. Carrillo, L.C.F. Ferreira, *The asymptotic behavior of subcritical dissipative quasi-geostrophic equations*, Nonlinearity 21 (5) (2008) 10011018.
- [10] J.-Y. Chemin, *Thormes d'unicit pour le systme de NavierStokes tridimensionnel*, J. Anal. Math. 77 (1999) 2750.
- [11] P. Constantin and J. Wu, *Behavior of solutions of 2D quasi-geostrophic equations*, SIAM J. Math. Anal. 30 (1999), no. 5, 937-948.
- [12] Y. Zhou, *Decay rate of higher order derivatives for solutions to the 2D dissipative quasi-geostrophic flows*, Discrete Contin. Dyn. Syst. 14 (2006), no. 3, 525-532.
- [13] P. Constantin, A. Majda, E. Tabak, *Formation of strong fronts in the 2D quasi-geostrophic thermal active scalar*, Nonlinearity 7.(1994) 1495-1533.

- [14] Wu, *Lower bounds for an integral involving fractional Laplacians and the generalized NavierStokes equations in Besov spaces*, Commun. Math. Phys. 263 (3) (2006) 803831.
- [15] H. Wang, Z. Zhang, *A Frequency Localized Maximum Principle Applied to the 2D Quasi-Geostrophic Equation*, Preprint.

DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, KING SAUD UNIVERSITY, RIYADH 11451,
KINGDOM OF SAUDI ARABIA

E-mail address: jbenameur@ksu.edu.sa

UNIVERSITY OF TUNIS EL MANAR, FACULTY OF SCIENCES OF TUNIS, LR03ES04, 2092 TUNIS, TUNISIA

E-mail address: moez.benhamed@hotmail.com